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Zeros of polynomials orthogonal over regular N-gons

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Abstract

We investigate the location of zeros of Bergman polynomials (orthogonal polynomials with respect to area measure) for regular N-gons in the plane. In particular, we prove two conjectures posed by Eiermann and Stahl. Furthermore, we give some consequences regarding the asymptotic behavior of such Bergman polynomials.

1. Introduction

Let $G\subset\mathbb{C}$ be a bounded Jordan domain. *Bergman polynomials* for G are algebraic polynomials $Q_n(z;G)$, $\deg Q_n = n$, in the complex variable z satisfying the orthogonality relation

$$\iint_G Q_m(z)\overline{Q_n(z)}\,dx\,dy = \delta_{m,n}, \quad z = x+iy. \tag{1}$$

These polynomials play an important role in different aspects of approximation theory. In particular, they have a close connection with the *interior* Riemann mapping function $\varphi_\zeta(z)$ for $\zeta\in G$, that is, the conformal map of G onto the unit disk

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$\{w\colon |w|<1\}$ satisfying $\varphi_\zeta(\zeta)=0,\ \varphi'_\zeta(\zeta)>0$. Namely,

$$\varphi'_\zeta(z)=\sqrt{\frac{\pi}{K(\zeta,\zeta)}}\,K(z,\zeta), \tag{2}$$

where $K(z,\zeta)$ is the *Bergman kernel*, which has the representation

$$K(z,\zeta)=\sum_{k=0}^{\infty}\overline{Q_k(\zeta)}Q_k(z). \tag{3}$$

We will be interested in the case when $G=G_N$ is the regular N-gon with vertices at $\omega_N^k,\ k=0,\dots,N-1$, where $\omega_N:=e^{2\pi i/N}$ is the first primitive Nth root of unity. More precisely, we will investigate the properties of zeros of $Q_n(z;G_N)$. Note that the convexity of G_N implies that all these zeros lie in the interior of G_N (for example, see [10, Theorem 2.2]). Furthermore, from symmetry arguments, if $n=Nl+j$, $0\leqslant j\leqslant N-1$, we deduce that

$$Q_n(z;G_N)=z^j q_l(z^N),\quad \deg q_l=l. \tag{4}$$

In [3], Eiermann and Stahl presented numerical results which led them to pose the following three conjectures.

(I) For $N=3,\ 4$, the zeros of all the Q_n's are located exactly on the "diagonals" $\Gamma_{k,N}$ of G_N:

$$\Gamma_{k,N}:=\{z\colon |z|<1,\ \arg z=2\pi k/N\}\cup\{0\},\quad k=\overline{1,N}.$$

However, for $N\geqslant 5$ there are zeros of Q_n's that are *not* on the $\Gamma_{k,N}$'s.

(II) For $N=3,\ 4$ and fixed $j\in\{0,\dots,N-1\}$, the real zeros of the Q_{Nl+j}'s interlace on $(0,1)$.

(III) For $N\geqslant 5$, the only points of the boundary ∂G_N of G_N that attract zeros of the Q_n's are its vertices, i.e., if Z_n denotes the set of zeros of Q_n, then

$$\left(\bigcap_{n=1}^{\infty}\overline{\bigcup_{m>n}Z_m}\right)\bigcap\partial G_N=\{\omega_N^k\}_{k=0}^{N-1}.$$

It was shown by Andrievskii and Blatt [1] that (III) is false for each $N\geqslant 5$ since, for such N, φ''_ζ blows up at the vertices of G_N. The following general result in this direction is due to Levin, Saff, and Stylianopoulos [7].

Theorem 1. *Let G be a bounded Jordan domain, Q_n the Bergman polynomials for G, and ν_n the normalized counting measure in the zeros of Q_n. Let $\mu_{\partial G}$ denote the equilibrium (Robin) measure for ∂G and let $\zeta\in G$. Then there exists a subsequence $\{n_k\}\subseteq\mathbb{N}$ such that*

$$\nu_{n_k}\xrightarrow{*}\mu_{\partial G}\quad \text{as } n_k\to\infty \tag{5}$$

if and only if the interior conformal mapping φ_ζ cannot be analytically continued to a domain $\tilde G\supset\bar G$.

The convergence in (5) is understood to hold in the weak-star topology.

Theorem 1 implies that, for $N \geqslant 5$, every point of ∂G_N attracts zeros of the Q_n's. However, if $N = 3$ or 4, Theorem 1 yields no information about the zeros of the Q_n's and, in this regard, it is a main purpose of the present note to show that (I) and (II) are true statements (see Corollaries 5 and 7).

2. Proof of Conjectures (I) and (II)

Regarding the orthogonality relation (1) for G_N, we first observe the following. Let $m = Nl + j,\ n = Nr + s,\ 0 \leqslant j, s \leqslant N - 1$, and suppose the polynomials P_m and P_n have the form

$$P_m(z) = z^j p_l(z^N), \quad P_n(z) = z^s p_r(z^N), \qquad \text{where } \deg p_l = l,\ \deg p_r = r.$$

Then, clearly, for any $A \subset \mathbb{C}$,

$$\iint_{\omega_N A} P_m(z)\overline{P_n(z)}\, dx\, dy = \omega_N^{j-s} \iint_A P_m(z)\overline{P_n(z)}\, dx\, dy. \tag{6}$$

Let Δ denote the triangle region with vertices at 0, 1, and ω_N. Then

$$G_N = \bigcup_{k=0}^{N-1} (\omega_N^k \Delta),$$

and using (6) we obtain

$$\begin{aligned}\iint_{G_N} P_m(z)\overline{P_n(z)}\, dx\, dy &= \sum_{k=0}^{N-1} \iint_{\omega_N^k \Delta} P_m(z)\overline{P_n(z)}\, dx\, dy \\ &= \sum_{k=0}^{N-1} \omega_N^{k(j-s)} \iint_\Delta P_m(z)\overline{P_n(z)}\, dx\, dy.\end{aligned}$$

Since

$$\sum_{k=0}^{N-1} (\omega_N^{j-s})^k = \begin{cases} 0, & \text{if } j \neq s, \\ N, & \text{if } j = s, \end{cases}$$

we conclude that

$$\iint_{G_N} P_m(z)\overline{P_n(z)}\, dx\, dy = 0 \quad \text{if } m \neq n \ (\mathrm{mod}\ N).$$

So, (1) carries useful information only for $m = n \ (\mathrm{mod}\ N)$. In this case, for $m \neq n$,

$$\iint_\Delta Q_m(z; G_N)\overline{Q_n(z; G_N)}\, dx\, dy = \frac{1}{N} \iint_{G_N} Q_m(z; G_N)\overline{Q_n(z; G_N)}\, dx\, dy = 0. \tag{7}$$

Next, we show that the orthogonality relation (7) implies that $Q_{Nl+j},\ 0 \leqslant j \leqslant N - 1$, restricted to $[0, 1]$, is orthogonal to a certain system of l polynomials that depend on N and j.

For $j = \overline{0, N-1}$ (i.e., $j = 0, 1, \ldots, N-1$) and $m = 0, 1, \ldots,$ let

$$f_{N,Nm+j}(x) := \mathrm{Im}[\omega_N^{j+1}(x - 1 - \overline{\omega_N})^{Nm+j+1}]. \tag{8}$$

Lemma 2. *For* $N \geqslant 3$, $j = \overline{0, N-1}$, *and* $l = 1, 2, \ldots,$

$$\int_0^1 Q_{Nl+j}(x; G_N) f_{N,Nm+j}(x)\, dx = 0 \quad \textit{for } m = \overline{0, l-1}. \tag{9}$$

Proof. In this proof we denote, for convenience, $\omega := \omega_N$ and $Q_n(z) := Q_n(z; G_N)$. Let $n > k$ and $n \pmod N = k \pmod N = j$. Using Green's formula (cf. [4, p. 10]) we get from (7) that

$$\int_\gamma Q_n(z) \bar{z}^{k+1}\, dz = 0,$$

where γ denotes the positively oriented boundary of the triangle Δ. If $\gamma_1 := [0, 1]$, $\gamma_2 := [1, \omega]$, and $\gamma_3 := [\omega, 0]$ denote the (oriented) sides of the triangle Δ, then we have on γ_1: $\bar{z} = z$, on γ_2: $\bar{z} = -\bar{\omega}(z-1) + 1$, on γ_3: $\bar{z} = \bar{\omega}^2 z$. Thus, using the Cauchy theorem and the fact that $Q_n(\omega\zeta) = \omega^k Q_n(\zeta)$, we get

$$\begin{aligned}
0 &= \int_\gamma Q_n(z)\bar{z}^{k+1}\, dz \\
&= \int_{\gamma_1} Q_n(z) z^{k+1}\, dz + \int_{\gamma_2} Q_n(z)(-\bar{\omega}(z-1)+1)^{k+1}\, dz + \int_{\gamma_3} Q_n(z)(\bar{\omega}^2 z)^{k+1}\, dz \\
&= \int_{\gamma_1} Q_n(z) z^{k+1}\, dz + \int_{-\gamma_1-\gamma_3} Q_n(z)(-\bar{\omega}(z-1)+1)^{k+1}\, dz \\
&\quad + \int_{\gamma_3} Q_n(z)(\bar{\omega}^2 z)^{k+1}\, dz \\
&= \int_{\gamma_1} Q_n(z)[z^{k+1} - (-\bar{\omega}(z-1)+1)^{k+1}]\, dz \\
&\quad + \int_{-\gamma_3} Q_n(z)[(-\bar{\omega}(z-1)+1)^{k+1} - (\bar{\omega}^2 z)^{k+1}]\, dz \\
&= \int_{\gamma_1} Q_n(z)[z^{k+1} - (-\bar{\omega}(z-1)+1)^{k+1}]\, dz \\
&\quad + \omega \int_{\gamma_1} Q_n(\omega\zeta)[(-\bar{\omega}(\omega\zeta-1)+1)^{k+1} - (\bar{\omega}\zeta)^{k+1}]\, d\zeta \\
&= (-1)^{k+1} \int_{\gamma_1} Q_n(z)[\omega^{j+1}(z-1-\bar{\omega})^{k+1} - \bar{\omega}^{j+1}(z-1-\omega)^{k+1}]\, dz.
\end{aligned}$$

All that remains is to note that, for real x,

$$\omega^{j+1}(x-1-\bar{\omega})^{k+1} - \bar{\omega}^{j+1}(x-1-\omega)^{k+1} = 2i f_{N,k}(x). \qquad \square$$

Remark 3. Note that, for any $N \geq 3$, $j \in \{0, 1, \dots, N-1\}$, and $l = 0, 1, \dots,$ the functions $f_{N,Nl+j}(x)$ have all real zeros and exactly l of them belong to $(0, 1)$. Indeed, the function

$$w = g(z) := \frac{\omega_N(z - 1 - \bar{\omega}_N)}{\bar{\omega}_N(z - 1 - \omega_N)}$$

maps the real axis $\operatorname{Im} z = 0$ onto the unit circle $|w| = 1$, and the image of $(0, 1)$ is the (shorter) open subarc γ_{ω_N} with endpoints 1 and ω_N. Now, in the w-plane, the equation $f_{N,Nl+j}(z) = 0$ is equivalent to

$$w^{Nl+j+1} = 1,$$

which has the roots $e^{2\pi i r/(Nl+j+1)}$, $r = \overline{0, Nl+j}$. One can easily check that l of these roots belong to γ_{ω_N}.

Lemma 4. *For* $N = 3$ *or* 4 *and fixed* j, $0 \leq j \leq N-1$, *the system* $\{f_{N,Nl+j}\}$, $l = 0, 1, 2, \dots,$ *is a Markov system on* $(0, 1)$, *i.e., any polynomial*

$$p_l(x) = \sum_{r=0}^{l} a_r f_{N,Nr+j}(x)$$

over this system that is not identically zero has at most l *zeros on* $(0, 1)$, $l = 0, 1, \dots.$ *Moreover, for each* $N \geq 5$ *and each* $j = \overline{0, N-1}$, *the system* $\{f_{N,Nl+j}\}_{l=0}^{\infty}$ *is not Markov on* $(0, 1)$.

Proof. We will prove the first part of the lemma by induction on l. First, for $l = 0$ the conclusion of the lemma holds thanks to Remark 3. Next, assume that, for some $l \geq 0$, the system $f_{N,Nr+j}(x)$, $r = 0, \dots, l$, is a Markov system on $(0, 1)$, and suppose, to the contrary, that a polynomial

$$p_{l+1}(x) = \sum_{r=0}^{l+1} a_r f_{N,Nr+j}(x), \quad a_{l+1} \neq 0,$$

has $l + 2$ zeros on $(0, 1)$. On differentiating N times, we obtain

$$\begin{aligned} p_{l+1}^{(N)}(x) &= a_0 f_{N,j}^{(N)}(x) + \sum_{r=1}^{l+1} a_r f_{N,Nr+j}^{(N)}(x) = \sum_{r=1}^{l+1} a_r c_{r,N} f_{N,N(r-1)+j}(x) \\ &= \sum_{r=0}^{l} b_r f_{N,Nr+j}(x) =: p_l(x), \end{aligned}$$

where $c_{r,N} := (Nr + j + 1)!/(Nr + j + 1 - N)!$ and $b_r := a_{r+1} c_{r+1,N}$. We shall show that p_l has at least $l + 1$ zeros in $(0, 1)$, which will yield the desired contradiction.

We remark that counting only *interior* zeros of a polynomial on $[0, 1]$, i.e., its zeros on $(0, 1)$, we can guarantee only one less zero on $(0, 1)$ for its derivative. At the same time, each *endpoint* zero of this polynomial gives an additional zero for the derivative on $(0,1)$. We claim that the polynomials

$$p_{l+1}^{(m)}(x), \quad m = \overline{0, N-1}, \tag{10}$$

have at least $N-1$ endpoint zeros in total, which would imply that p_l has at least $l+1$ zeros on $(0,1)$.

For fixed $j = \overline{0, N-1}$, let us first investigate the endpoint zeros of $f_{N,Nr+j}^{(m)}(x)$, $r = 0, 1, \ldots$. Clearly,

$$f_{N,Nr+j}^{(m)}(x) = c_{r,m} \operatorname{Im}[\omega_N^{j+1}(x - 1 - \bar{\omega}_N)^{Nr+j+1-m}],$$
$$c_{r,m} := (Nr + j + 1)!/(Nr + j + 1 - m)!.$$

So, after some algebra, we get

$$f_{N,Nr+j}^{(m)}(0) = (-1)^{Nr+j+1-m+r} c_{r,m} \left(2\cos\frac{\pi}{N}\right)^{Nr+j+1-m} \sin\left(\frac{j+1+m}{N}\pi\right) \tag{11}$$

and

$$f_{N,Nr+j}^{(m)}(1) = (-1)^{Nr+j+1-m} c_{r,m} \operatorname{Im}(\omega_N^m) = (-1)^{Nr+j+1-m} c_{r,m} \sin\left(\frac{2\pi m}{N}\right). \tag{12}$$

For $N \geqslant 3$, $f_{N,Nr+j}^{(m)}(0) = 0$ if and only if

$$\sin\left(\frac{j+1+m}{N}\pi\right) = 0 \tag{13}$$

(for $r = 0$ and $m > j$, obviously $f_{N,j}^{(m)}(x) \equiv 0$). But $0 \leqslant j \leqslant N-1$, $0 \leqslant m \leqslant N-1$ and hence $1 \leqslant j + 1 + m \leqslant 2N - 1$. Thus, (13) holds only in the case $j + m = N - 1$ regardless of r, i.e., for $m = N - 1 - j$ and any $r = 0, 1, \ldots, f_{N,Nr+j}^{(m)}(0) = 0$ and, therefore,

$$p_{l+1}^{(m)}(0) = 0 \quad \text{if } m = N - 1 - j.$$

Thus, to establish the claim it is enough to show that $N-2$ polynomials in (10) have a zero at $x = 1$, for which, according to (12), a sufficient condition is that

$$\sin\left(\frac{2\pi m}{N}\right) = 0 \tag{14}$$

for $N-2$ values of $m \in \{0, \ldots, N-1\}$. But $0 \leqslant 2m/N < 2$ and so there are at most two values of m for which (14) is true. So, we should restrict ourselves to the case $N \leqslant 4$. For $N = 3$, we need just one zero at $x = 1$, and this happens when $m = 0$. For $N = 4$, the required two zeros occur when $m = 0$ and $m = 2$. This completes the proof of the first part of the lemma.

Now we consider the case when $N \geqslant 5$. As in the proof of Corollary 5 below, the fact that, for some $j \in \{0, \ldots, N-1\}$, the system $\{f_{N,Nl+j}\}$, $l = 0, 1, \ldots$, is a Markov system on $(0,1)$ implies that all the zeros of the $Q_{Nl+j}(z; G_N)$'s, $l = 0, 1, \ldots$, lie on the rays $\Gamma_{k,N}$, $k = \overline{1, N}$. Since, for such N, $\varphi_\zeta(z)$ cannot be extended analytically to a larger region, using Theorem 1 we conclude that $\{f_{N,Nl+j}\}$, $l = 0, 1, \ldots$, is not

Markov at least for *some* $j \in \{0, \dots, N-1\}$. The fact that this system is not Markov for *every* $j \in \{0, \dots, N-1\}$ requires additional arguments, and we proceed as follows.

Using the representations (2)–(4) we get, for any $\zeta \in G_N$,

$$\varphi'_\zeta(z) = g_0(z^N, \zeta) + zg_1(z^N, \zeta) + \cdots + z^{N-1}g_{N-1}(z^N, \zeta), \tag{15}$$

where

$$g_j(z^N, \zeta) := \sqrt{\frac{\pi}{K(\zeta, \zeta)}} \sum_{k=0}^{\infty} \overline{Q_{Nk+j}(\zeta)} \frac{Q_{Nk+j}(z)}{z^j}.$$

In particular, for $\zeta = 0$, we have $Q_{Nk+j}(0) = 0$ for $j = \overline{1, N-1}$, and so

$$\varphi'_0(z) = g_0(z^N, 0) = \sqrt{\pi/K(0,0)} \sum_{l=0}^{\infty} \overline{Q_{Nl}(0)}\, Q_{Nl}(z).$$

The regularity of the Lebesgue measure implies (cf. [8, Lemma 4.3]) that for the sup norm $||\cdot||_{G_N}$ on G_N,

$$\lim_{n\to\infty} ||Q_n||_{G_N}^{1/n} = 1. \tag{16}$$

Since $\varphi'_0(z)$ does not have an analytic extension to a domain $\tilde{G} \supset \bar{G}_N$, it follows that

$$\limsup_{l\to\infty} |Q_{Nl}(0)|^{1/Nl} = 1. \tag{17}$$

As in the proof of Theorem 1.1 in [7] we invoke Theorem III.4.1 in [11] to conclude that, for some subsequence $\{l_k\}_{k=1}^{\infty}$, the normalized counting measures v_{Nl_k} of the zeros of $Q_{Nl_k}(z; G_N)$ satisfy

$$v_{Nl_k} \xrightarrow{*} \mu_{\partial G_N} \quad \text{as } l_k \to \infty. \tag{18}$$

Consequently, $\{f_{N,Nl+j}\}$, $l = 0, 1, \dots,$ is not Markov for $j = 0$.

Next we observe that, for any integer k,

$$\varphi_0(\omega_N^k z) = \omega_N^k \varphi_0(z) \quad \text{and} \quad \varphi'_0(\omega_N^k z) = \varphi'_0(z). \tag{19}$$

Also note that, for any $\zeta \in G_N$,

$$\varphi_\zeta(z) = \lambda \frac{\varphi_0(z) - \varphi_0(\zeta)}{1 - \overline{\varphi_0(\zeta)}\varphi_0(z)}, \quad \varphi'_\zeta(z) = \lambda \frac{\varphi'_0(z)(1 - |\varphi_0(\zeta)|^2)}{(1 - \overline{\varphi_0(\zeta)}\varphi_0(z))^2}, \quad |\lambda| = 1. \tag{20}$$

Setting $\mathscr{F}_0(z, \zeta) := \varphi'_\zeta(z)$ and, for $j = 1, \dots, N-1$,

$$\mathscr{F}_j(z, \zeta) := \frac{\mathscr{F}_{j-1}(z, \zeta) - g_{j-1}(z^N, \zeta)}{z},$$

and using (15), (19), and (20), after some algebra we get

$$
\begin{aligned}
Ng_j(z^N,\zeta) &= \sum_{k=0}^{N-1} \mathscr{F}_j(\omega_N^k z) \\
&= N\lambda(1-|\varphi_0(\zeta)|^2)\varphi_0'(z)\left(\frac{\overline{\varphi_0(\zeta)}\varphi_0(z)}{z}\right)^j \\
&\quad \times \frac{j+1+(N-j-1)(\overline{\varphi_0(\zeta)}\varphi_0(z))^N}{(1-(\overline{\varphi_0(\zeta)}\varphi_0(z))^N)^2}.
\end{aligned}
$$

On differentiating this equation and using the facts that $\varphi_0''(z)\to\infty$, $\varphi_0'(z)$ is bounded, and $\varphi_0(z)\to 1$ as $z\to 1$, $z\in G_N$, one easily concludes that $g_j(z^N,\zeta)$ cannot be extended analytically to a larger domain for some $\zeta\neq 0$ in G_N. Taking into account this fact, we now repeat the argument used for $j=0$ to conclude from (16) the analogs of (17) and (18); that is,

$$\limsup_{l\to\infty} |Q_{Nl+j}(\zeta)|^{1/(Nl+j)} = 1$$

and, for some subsequence $\{l_k\}_{k=1}^\infty$ that depends on j,

$$\nu_{Nl_k+j} \xrightarrow{*} \mu_{\partial G_N} \quad \text{as } l_k\to\infty. \tag{21}$$

Therefore, $\{f_{N,Nl+j}\}$, $l=0,1,\dots,$ is not Markov for every $j=0,\dots,N-1$. $\square$

We remark that (21) provides some new information regarding the asymptotic behavior of the zeros of $Q_n(z;G_N)$ for the cases $N\geqslant 5$.

Corollary 5. *For $N=3$ or 4 and $j=\overline{0,N-1}$, the polynomials $Q_{Nl+j}(x;G_N)$, $l=0,1,\dots,$ have exactly l simple zeros on $(0,1)$. Consequently, all zeros of $Q_{Nl+j}(z;G_N)$ lie on the rays $\Gamma_{k,N}$, $k=\overline{1,N}$.*

Proof. Using Lemma 4 and the orthogonality relation (9), we conclude from well-known arguments originally given by Kellog [6] (see also [9, Proposition 3.1]) that Q_{Nl+j} has at least l sign changes on $(0,1)$. But it follows from the symmetry property (4) that Q_{Nl+j} cannot have more than l zeros on $(0,1)$. $\square$

Next, for fixed j, we establish the interlacing property of zeros of the Q_{Nl+j}'s. This property is a consequence of the following general statement.

Lemma 6. *Let $\{g_k(t)\}_{k=0}^\infty$ be a Markov system of continuous functions on (a,b), and suppose that polynomials $P_n(t)$, $\deg P_n\leqslant n$, $n=1,2,\dots,$ are orthogonal to $g_k(t)$, $k=\overline{0,n-1}$, on (a,b), i.e.,*

$$\int_a^b P_n(t)g_k(t)\,dt = 0. \tag{22}$$

Then between any two consecutive zeros of $P_n(t)$ *on* (a, b) *there is a (unique) zero of* $P_{n-1}(t)$.

Although similar results are known (cf. [5]) for the case when the P_n's are in the span of the g_k's, the authors could not find the needed form in the literature, so we provide a simple proof.

Proof. First of all, we note that all zeros of $P_n(t),\ n = 1, 2, \ldots,$ are simple, and lie on (a, b). Suppose now, to the contrary, that α and β are two consecutive zeros of $P_{n+1}(t)$ and $P_n(t)$ has no zeros on (α, β). We can assume without loss of generality that $P_n(t) \geqslant 0$ and $P_{n+1}(t) \geqslant 0$ on $[\alpha, \beta]$. Consider the polynomial

$$R_{n+1}(t) := cP_n(t) - P_{n+1}(t),$$

where the constant $c > 0$ is chosen as follows:

(i) if $P_n(t) = 0$ either at α or at β, denote this point by t^* and set
$$c := \frac{P'_{n+1}(t^*)}{P'_n(t^*)};$$
clearly, $R_{n+1}(t)$ has a zero at t^* of multiplicity at least two.

(ii) otherwise, $P_n(t) > 0$ on $[\alpha, \beta]$ and, with
$$c := \min\{C\colon\ C \geqslant 0,\ \ CP_n(t) - P_{n+1}((t)) \geqslant 0 \ \text{on}\ [\alpha, \beta]\},$$
the polynomial $R_{n+1}(t)$ has a zero $t^* \in (\alpha, \beta)$ of even multiplicity.

With such a choice for c, the polynomial $R_{n+1}(t)/(t - t^*)^2$ has no more than $n - 1$ zeros on (a, b), and so no more than $n - 1$ sign changes. Hence, $R_{n+1}(t)$ has no more than $n - 1$ sign changes on (a, b), and one can find a function

$$G_n(t) = \sum_{s=0}^{n-1} a_s g_s(t)$$

over the system $\{g_s\}_{s=0}^{n-1}$ such that the product $R_{n+1}(t)G_n(t)$ is nonnegative on (a, b). On the other hand, the orthogonality relation (22) gives

$$\int_a^b R_{n+1}(t)G_n(t)\, dt = 0.$$

This implies that either $R_{n+1}(t)$ or $G_n(t)$ must be identically zero on (a, b), which is impossible. $\square$

Corollary 7. *For* $N = 3$ *or* 4 *and fixed* $j \in \{0, \ldots, N-1\}$, *between any two consecutive zeros of* $Q_{Nl+j}(x; G_N),\ l = 2, 3, \ldots,$ *on* $(0, 1)$ *there is a (unique) zero of* $Q_{N(l-1)+j}(x; G_N)$.

Proof. We apply Lemma 6 to the polynomials $P_l(t) := q_l(t),\ l = 0, 1, \ldots,$ with $q_l(t)$ defined in (4) and the system $g_k(t) := t^{(j+1-N)/N} f_{N,Nk+j}(t^{1/N}),\ k = 0, 1, \ldots,$ with $f_{N,Nk+j}(x)$ given by (8), which, by Lemma 4, is a Markov system on $(0,1)$ (since j is fixed). The orthogonality relation (22) follows immediately from (9) with the substitution $t = x^N$. $\square$

Corollaries 5 and 7 establish the truth of assertions (I) and (II).

Let $\Phi_N(z)$ denote the *exterior* Riemann mapping function for G_N, i.e., $\Phi_N\colon \bar{\mathbb{C}} \backslash \bar{G}_N \mapsto \{|w| > 1\},\ \Phi_N(\infty) = \infty,\ \Phi_N'(\infty) > 0$. Using, for each side of G_N, the Schwarz reflection principle, we can extend Φ_N to a function $\tilde{\Phi}_N(z)$ that is analytic and one-to-one in $\mathbb{C}\backslash(\bigcup_{k=1}^N \bar{\Gamma}_{k,N})$.

Corollary 8. *For* $N = 3$ *or* 4,

$$\lim_{n\to\infty} Q_n(z; G_N)^{1/n} = \tilde{\Phi}_N(z)$$

locally uniformly in $\mathbb{C}\backslash(\bigcup_{k=1}^N \bar{\Gamma}_{k,N})$, *where* $x^{1/n}$ *denotes the branch that is positive for* $x > 0$.

Proof. Indeed, the fact that all the zeros of $Q_n(z; G_N)$'s are located on the rays $\Gamma_{k,N},\ k = \overline{1, N}$, makes it possible to define single-valued analytic branches of the functions $Q_n(z; G_N)^{1/n},\ n = 1, 2, \ldots,$ in the domain $\mathbb{C}\backslash(\bigcup_{k=1}^N \bar{\Gamma}_{k,N})$. These functions form a normal family in this domain and, moreover, it is well-known [12, Chapter 3] that

$$\lim_{n\to\infty} Q_n(z; G_N)^{1/n} = \Phi_N(z)$$

locally uniformly in $\mathbb{C}\backslash\bar{G}_N$. Thus, the assertion follows from standard uniqueness theorems. $\square$

Theorem 9. *For* $N = 3$ *or* 4, *let* $\lambda_{N,j}^{(l)}$ *be the normalized counting measure of the zeros of* $Q_{Nl+j}(z)$ *that lie in* $(0,1)$, *i.e.*,

$$\lambda_{N,j}^{(l)} = \frac{1}{l} \sum_{\substack{x \in Z_{Nl+j} \\ x > 0}} \boldsymbol{\delta}_x,$$

where $\boldsymbol{\delta}_x$ *is the unit point mass at* x. *Then there exists a measure* μ_N *such that for each* $j = \overline{0, N-1}$

$$\lambda_{N,j}^{(l)} \xrightarrow{*} \mu_N \quad \text{as } l \to \infty.$$

Moreover, μ_N *is the unique measure supported on* $[0,1]$ *that satisfies the equation*

$$\ln|\tilde{\Phi}_N(z)| = \frac{1}{N} \int \ln|z^N - x^N|\, d\mu_N(x) + \ln\frac{1}{c_N} \tag{23}$$

for all $z \notin \bigcup_{k=1}^N \bar{\Gamma}_{k,N}$, *where* c_N *is the logarithmic capacity of* G_N.

Proof. For any positive measure λ let $U(z;\lambda)$ denote its logarithmic potential

$$U(z;\lambda) := \int \ln \frac{1}{|z-t|}\, d\lambda(t).$$

First we observe that the regularity of the Lebesgue measure over G_N implies that for each $j = \overline{0, N-1}$

$$U(z; \nu_{Nl+j}) \to U(z; \mu_{\partial G_N}), \quad z \notin \bar{G}_N, \tag{24}$$

where ν_{Nl+j} is the normalized counting measure of Z_{Nl+j}, the set of all zeros of Q_{Nl+j}. Note by symmetry, that

$$\nu_{Nl+j}(\cdot) = \frac{1}{Nl+j}\left\{ j\boldsymbol{\delta}_0(\cdot) + l \sum_{k=0}^{N-1} \lambda_{N,j}^{(l)}(\omega_N^k \cdot) \right\}.$$

Hence from (24) it follows that if λ is any limit measure of $\{\lambda_{N,j}^{(l)}\}_{l=0}^{\infty}$, then

$$U(z; \mu_{\partial G_N}) = U\left(z; \frac{1}{N} \sum_{k=0}^{N-1} \lambda(\omega_N^k \cdot)\right) \quad \text{for } z \notin \bar{G}_N\,.$$

Writing

$$U(z; \mu_{\partial G_N}) = \ln \frac{1}{c_N} - \ln|\Phi_N(z)|,$$

we obtain (23) for $z \notin \bar{G}_N$ and $\mu_N = \lambda$. Since $\operatorname{supp}(\lambda) \subset [0,1]$ Eq. (23) holds by harmonic continuation for all $z \in \mathbb{C} \setminus \bigcup_{k=1}^{N} \bar{\Gamma}_{k,N}$.

Finally, we can use the unicity theorem for logarithmic potentials (cf. [11, Theorem II.2.1]) to deduce that (23) uniquely determines the measure μ_N and so every limit measure λ must equal μ_N. □

We remark that for convex domains G, results concerning the asymptotic behavior of the *balayages* (to the boundary of G) of the zeros of the Bergman polynomials were obtained in [2].

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